

Generalised geometry, eleven dimensions and E_{11}

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Abstract

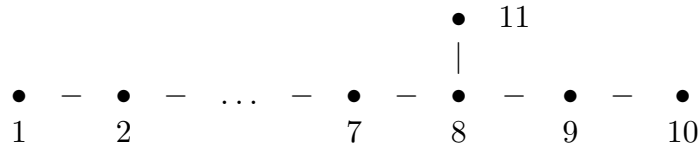
We construct the non-linear realisation of E_{11} and its first fundamental representation in eleven dimensions at low levels. The fields depend on the usual coordinates of space-time as well as two form and five form coordinates. We derive the terms in the dynamics that contain the three form and six form fields and show that when we restrict their field dependence to be only on the usual space-time we recover the correct self-duality relation. Should this result generalise to the gravity fields then the non-linear realisation is an extension of the maximal supergravity theory, as previously conjectured. We also comment on the connections between the different approaches to generalised geometry.

1. Introduction

It has been realised for more than a quarter of a century that there exists no fundamental theory of strings. The maximal supergravity theories are the complete low energy effective actions for the type II string theories and these have provided a source of certainty. However, the study of these theories has also led to the realisation that a fundamental theory of strings must include branes, but we have really no idea how to describe many of the properties of branes. One way forward may be to try to guess the symmetries of the underlying theory. Some time ago, and with this approach in mind, it was conjectured that a non-linear realisation of the Kac-Moody algebra E_{11} was an extension of the maximal supergravity theories [1]. The different maximal theories emerge from the different possible decompositions of E_{11} into the sub-algebras that arise from deleting the different possible nodes in the E_{11} Dynkin diagram [2,3,4,5,6,7]. In the early papers on E_{11} space-time was introduced into the non-linear realisation by adjoining to the E_{11} algebra the space-time translations. It was understood at the time that this was an adhoc step and in 2003 it was proposed that one should consider the non-linear realisation of the semi-direct product of E_{11} and its first fundamental representation l_1 denoted $E_{11} \otimes_s l_1$ [8]. The latter contains the generators P_a , Z^{ab} , $Z^{a_1 \dots a_5}$, $Z^{a_1 \dots a_7, b}$ as well as an infinite number of other objects. In the non-linear realisation this leads to a generalised space-time with coordinates x^a , x_{ab} , $x_{a_1 \dots a_5}$, $x_{a_1 \dots a_7, b}$, \dots [8]. There is very good evidence that the l_1 representation contains all the brane charges [8,9,10,11,3] and so there is a one to one relationship between the coordinates of the generalised space-time and brane charges. To appreciate this proposal one has to understand what is a non-linear realisation which is in this case not the same as what is often called a sigma model. Such non-linear realisations were given in the papers on E_{11} , for example [12,1,5], and an early application was to formulate gravity as a non-linear realisation [14]. The non-linear realisation of $E_{11} \otimes_s l_1$ not only introduces a generalised space-time but also a generalised vielbein and so a corresponding geometry. In particular this non-linear realisation was used to derive almost all of the features of the five dimensional gauged maximal supergravities [5]. However, there has not been a systematic attempt to construct the non-linear realisation of $E_{11} \otimes_s l_1$. In this paper we will construct the dynamics of the $E_{11} \otimes_s l_1$ at lowest level keeping the first few coordinates of the generalised space-time, that is the coordinates x^a , x_{ab} and $x_{a_1 \dots a_5}$ and the three form and six form fields.

2. A review of E_{11} and its l_1 representation

In this section we review some of the technical aspects of E_{11} and its first fundamental representation l_1 which will be required to construct the non-linear realisation and so the dynamics. The E_{11} algebra, like any Kac-Moody algebra, is just the multiple commutator of the Chevalley generators subject to the Serre relations. The Dynkin diagram of E_{11} is given by



The eleven dimensional theory emerges from the E_{11} non-linear realisation if we delete

node eleven and decompose the E_{11} algebra in terms of the remaining A_{10} subalgebra; that is decompose the adjoint representation of E_{11} in terms representations of A_{10} . The generators can be listed according to a level and those of positive level are given by [1,13]

$$K^a_b, R^{a_1 a_2 a_3}, R^{a_1 a_2 \dots a_6} \text{ and } R^{a_1 a_2 \dots a_8, b} \quad (2.1)$$

at levels zero, one, two and three respectively. The generators at level zero are those of $GL(11)$ and are responsible in the non-linear realisation for eleven dimensional gravity. The generator $R^{a_1 a_2 \dots a_8, b}$ obeys the condition $R^{[a_1 a_2 \dots a_8, b]} = 0$.

The corresponding negative level generators are given by

$$R_{a_1 a_2 a_3}, R_{a_1 a_2 \dots a_6} \text{ and } R_{a_1 a_2 \dots a_8, b}, \quad (2.2)$$

at levels -1, -2,-3 with the last generator satisfying an analogous constraint.

From the mathematical viewpoint the E_{11} algebra is just the multiple commutators of the Chevalley generators subject to the Serre relations. However, it turns out that the Chevalley generators are contained in the generators K^a_b , $R^{a_1 a_2 a_3}$ and $R_{a_1 a_2 a_3}$ and so the E_{11} algebra is found by taking the multiple commutators of these generators and at low levels it suffices to just impose the Jacobi identities on the algebra formed from the generators listed above. The commutators must preserve the level and so on the right-hand side of the commutators we can only write all possible terms that preserve the level. We can then implement the Jacobi identities. The level is plus (minus) the number of times the positive (negative) root Chevalley generators associated with node eleven occur in the multiple commutator that creates the generator. However, this is just the same as plus (minus) the number of times the generator $R^{a_1 a_2 a_3}$ ($R_{a_1 a_2 a_3}$) occurs in the multiple commutator.

The generators of $GL(11)$ obey the algebra

$$[K^a_b, K^c_d] = \delta_b^c K^a_d - \delta_d^a K^c_b, \quad (2.3)$$

By construction the generators in equations (2.1) and (2.2) are representations of $GL(11)$ and so their commutators with the K^a_b generators are given by

$$[K^a_b, R^{c_1 \dots c_6}] = \delta_b^{c_1} R^{a c_2 \dots c_6} + \dots, [K^a_b, R^{c_1 \dots c_3}] = \delta_b^{c_1} R^{a c_2 c_3} + \dots, \quad (2.4)$$

$$[K^a_b, R^{c_1 \dots c_8, d}] = (\delta_b^{c_1} R^{a c_2 \dots c_8, d} + \dots) + \delta_b^d R^{c_1 \dots c_8, a}. \quad (2.5)$$

where $+\dots$ means the appropriate anti-symmetrisation. The corresponding relations for the negative level generators are

$$[K^a_b, R_{c_1 \dots c_3}] = -\delta_{c_1}^a R_{b c_2 c_3} - \dots, [K^a_b, R_{c_1 \dots c_6}] = -\delta_{c_1}^a R_{b c_2 \dots c_6} - \dots, \quad (2.6)$$

$$[K^a_b, R_{c_1 \dots c_8, d}] = -(\delta_{c_1}^a R_{b c_2 \dots c_8, d} + \dots) - \delta_d^a R_{c_1 \dots c_8, b}. \quad (2.7)$$

The rest of the E_{11} algebra can be found by remembering that the commutators preserve the level, writing the most general possibility on the right hand side of the commutator, and enforcing the Jacobi identities. For the positive level generators we find that [1]

$$[R^{c_1 \dots c_3}, R^{c_4 \dots c_6}] = 2R^{c_1 \dots c_6}, \quad [R^{a_1 \dots a_6}, R^{b_1 \dots b_3}] = 3R^{a_1 \dots a_6 [b_1 b_2, b_3]}, \quad (2.8)$$

and for the negative root generators

$$[R_{c_1 \dots c_3}, R_{c_4 \dots c_6}] = 2R_{c_1 \dots c_6}, \quad [R_{a_1 \dots a_6}, R_{b_1 \dots b_3}] = 3R_{a_1 \dots a_6 [b_1 b_2, b_3]}, \quad (2.9)$$

Finally, the commutation relations between the positive and negative generators of up to level four are given by [8]

$$\begin{aligned} [R^{a_1 \dots a_3}, R_{b_1 \dots b_3}] &= 18\delta_{[b_1 b_2}^{[a_1 a_2} K^{a_3]}_{b_3]} - 2\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} D, \quad [R_{b_1 \dots b_3}, R^{a_1 \dots a_6}] = \frac{5!}{2}\delta_{b_1 b_2 b_3}^{[a_1 a_2 a_3} R^{a_4 a_5 a_6]} \\ [R^{a_1 \dots a_6}, R_{b_1 \dots b_6}] &= -5!.3.3\delta_{[b_1 \dots b_5}^{[a_1 \dots a_5} K^{a_6]}_{b_6]} + 5!\delta_{b_1 \dots b_6}^{a_1 \dots a_6} D, \\ [R_{a_1 \dots a_3}, R^{b_1 \dots b_8, c}] &= 8.7.2(\delta_{[a_1 a_2 a_3}^{[b_1 b_2 b_3} R^{b_4 \dots b_8]c} - \delta_{[a_1 a_2 a_3}^{[b_1 b_2 | c]} R^{b_3 \dots b_8]}) \\ [R_{a_1 \dots a_6}, R^{b_1 \dots b_8, c}] &= \frac{7!.2}{3}(\delta_{[a_1 \dots a_6}^{[b_1 \dots b_6} R^{b_7 b_8]c} - \delta_{[a_1 \dots a_6}^{c[b_1 \dots b_5} R^{b_6 b_7 b_8]}) \end{aligned} \quad (2.10)$$

where $D = \sum_b K^b_b$, $\delta_{b_1 b_2}^{a_1 a_2} = \frac{1}{2}(\delta_{b_1}^{a_1} \delta_{b_2}^{a_2} - \delta_{b_1}^{a_2} \delta_{b_2}^{a_1}) = \delta_{b_1}^{[a_1} \delta_{b_2}^{a_2]}$ with similar formulae when more indices are involved.

A non-linear realisation is defined by a choice of an algebra together with a subalgebra. For us the subalgebra is chosen to be the one that is invariant under the Cartan involution I_c . This is an involution, that is I_c^2 is the identity operator, and an automorphism of the algebra, that is $I_c(AB) = I_c(A)I_c(B)$, which acts on the generators given above as

$$\begin{aligned} I_c(K^a_b) &= -\eta^{ac} \eta_{bd} K^d_c, \quad I_c(R^{a_1 a_2 a_3}) = -\eta^{a_1 b_1} \eta^{a_2 b_2} \eta^{a_3 b_3} R_{b_1 b_2 b_3}, \\ I_c(R^{a_1 \dots a_6}) &= \eta^{a_1 b_1} \dots \eta^{a_6 b_6} R_{b_1 \dots b_6}, \quad I_c(R^{a_1 \dots a_8, c}) = -\eta^{a_1 b_1} \dots \eta^{a_6 b_6} \eta^{cd} R_{b_1 \dots b_8, d} \end{aligned} \quad (2.11)$$

Its more fundamental definition in terms of its action on the Chevalley generators can be found in for example [1]. In fact, we have modified the usual definition to take account of the Minkowski rather than the Euclidean signature. The sub-algebra invariant under the Cartan involution is generated at low levels by

$$J_{ab} = K^c_b \eta_{ac} - K^c_a \eta_{bc}, \quad S_{a_1 a_2 a_3} = R^{b_1 b_2 b_3} \eta_{b_1 a_1} \eta_{b_2 a_2} \eta_{b_3 a_3} - R_{a_1 a_2 a_3}, \quad (2.12)$$

$$S_{a_1 \dots a_6} = R^{b_1 \dots b_6} \eta_{b_1 a_1} \dots \eta_{b_6 a_6} + R_{a_1 \dots a_6} \quad (2.13)$$

$$S_{a_1 \dots a_8, c} = R^{b_1 \dots b_8, b} \eta_{b_1 a_1} \dots \eta_{b_8 a_8} \eta_{bc} - R_{a_1 \dots a_8, c}. \quad (2.14)$$

where η_{ab} is the metric of Minkowski space-time. The generators J_{ab} are those of the Lorentz algebra $SO(1,10)$ and their commutators with the other generators just express the fact that they belong to a representation of the Lorentz algebra. The $S_{a_1 a_2 a_3}$ and $S_{a_1 \dots a_6}$ generators obey the commutators [8]

$$[S^{a_1 a_2 a_3}, S_{b_1 b_2 b_3}] = -18\delta_{[b_1 b_2}^{[a_1 a_2} J^{a_3]}_{b_3]} + 2S^{a_1 a_2 a_3}_{b_1 b_2 b_3} \quad (2.15)$$

$$[S_{a_1 a_2 a_3}, S^{b_1 \dots b_6}] = -\frac{5!}{2} \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} S^{b_4 b_5 b_6]} - 3 S^{b_1 \dots b_6}{}_{[a_1 a_2, a_3]} \quad (2.16)$$

The non-linear realisation of interest to us also includes generators in the fundamental representation of E_{11} , denoted l_1 ; by definition this representation has highest weight Λ_1 which obeys $(\Lambda_1, \alpha_a) = \delta_{a,1}$. Decomposed into representations of $GL(11)$ the l_1 representation contains, at low levels, the generators [8,9]

$$P_a, Z^{ab}, Z^{a_1 \dots a_5}, Z^{a_1 \dots a_7, b}, Z^{a_1 \dots a_8}, Z^{b_1 b_2 b_3, a_1 \dots a_8}, \dots \quad (2.17)$$

at levels 0,1,2,3,3 and 4 respectively. Here P_a is the generator of space-time translations and the next two generators can be identified with the central charges in the supersymmetry algebra. The commutators of the low level generators of the l_1 representation with $R^{a_1 a_2 a_3}$ are determined up to constants by demanding that the levels match and so we can take [8]

$$[R^{a_1 a_2 a_3}, P_b] = 3\delta_b^{[a_1} Z^{a_2 a_3]}, \quad [R^{a_1 a_2 a_3}, Z^{b_1 b_2}] = Z^{a_1 a_2 a_3 b_1 b_2},$$

$$[R^{a_1 a_2 a_3}, Z^{b_1 \dots b_5}] = Z^{b_1 \dots b_5 [a_1 a_2, a_3]} + Z^{b_1 \dots b_5 a_1 a_2 a_3} \quad (2.18)$$

These equations define the normalisation of these generators of the l_1 representation.

We will be interested in constructing the non-linear realisation of the semi-direct product of the E_{11} algebra with its l_1 representation, denoted $E_{11} \otimes_s l_1$. In this algebra the commutators of the generators of E_{11} with themselves obey the same algebra as above that is equations (2.4) to (2.10). We will take the generators of the l_1 representation to commute with themselves. The commutators between the generators of E_{11} and those of the l_1 representation express the fact that they are a representation of E_{11} and this is enforced by demanding the Jacobi identity involving two E_{11} generators and one l_1 generator. The construction is essentially the same as that for the Poincare group where the Lorentz group L plays the role of E_{11} and the space-time translations T the role of the l_1 representation; that is we can write the Poincare group as $L \otimes_s T$.

As we have decomposed the l_1 representation into representations of $GL(11)$, the commutators of these generators with those of the l_1 representations are given [8]

$$[K^a{}_b, P_c] = -\delta_c^a P_b + \frac{1}{2} \delta_b^a P_c, \quad [K^a{}_b, Z^{c_1 c_2}] = 2\delta_b^{[c_1} Z^{a|c_2]} + \frac{1}{2} \delta_b^a Z^{c_1 c_2},$$

$$[K^a{}_b, Z^{c_1 \dots c_5}] = 5\delta_b^{[c_1} Z^{a|c_2 \dots c_5]} + \frac{1}{2} \delta_b^a Z^{c_1 \dots c_5} \quad (2.19)$$

The term with the factor of $\frac{1}{2}$ plays an important role in many applications of E_{11} and it follows from the fact that the l_1 is a highest weight representation of E_{11} [8]. Strictly speaking it is actually a lowest weight representation as usually defined. We also find using the Jacobi identities and equations (2.8) and (2.18) that

$$[R^{a_1 \dots a_6}, P_b] = -3\delta_b^{[a_1} Z^{\dots a_6]}, \quad [R^{a_1 \dots a_6}, Z^{b_1 b_2}] = -Z^{b_1 b_2 [a_1 \dots a_5, a_6]} - Z^{b_1 b_2 a_1 \dots a_6}, \quad (2.20)$$

The commutators with the negative root generators are given by

$$[R_{a_1 a_2 a_3}, P_b] = 0, [R_{a_1 a_2 a_3}, Z^{b_1 b_2}] = 6\delta_{[a_1 a_2}^{b_1 b_2} P_{a_3]}, [R_{a_1 a_2 a_3}, Z^{b_1 \dots b_5}] = \frac{5!}{2} \delta_{a_1 a_2 a_3}^{[b_1 b_2 b_3} Z^{b_4 b_5]} \quad (2.21)$$

The first equation, just follows from the fact that the l_1 representation is a highest weight representation. and the subsequent equations follow by using the equation (2.10) and the Jacobi identities.

To conclude this section we now also give the commutation relations between the generators of the Cartan involution invariant subalgebra, given in equations (2.12-2.14), and the generators of the l_1 representation [8]

$$[S^{a_1 a_2 a_3}, P_b] = 3\delta_b^{[a_1} Z^{a_1 a_3]}, [S_{a_1 a_2 a_3}, Z^{b_1 b_2}] = Z_{a_1 a_2 a_3}{}^{b_1 b_2} - 6\delta_{[a_1 a_2}^{b_1 b_2} P_{a_3]},$$

$$[S_{a_1 a_2 a_3}, Z^{b_1 \dots b_5}] = Z^{b_1 \dots b_5}{}_{[a_1 a_2, a_3]} + Z^{b_1 \dots b_5}{}_{a_1 a_2 a_3} - \frac{5!}{2} \delta_{a_1 a_2 a_3}^{[b_1 \dots b_3} Z^{b_4 b_5]} \quad (2.24)$$

3. The non-linear realisation of $E_{11} \otimes_s l_1$

It was conjectured [8] that the non-linear realisation of $E_{11} \otimes_s l_1$ was an extension of the equations of motion of eleven dimensional supergravity. Put another way, it states that eleven dimensional supergravity was contained in the non-linear realisation of $E_{11} \otimes_s l_1$ at low levels. At higher levels one finds not only an infinite number of new fields coming from E_{11} , but also all the fields depend on a generalised space-time encoded in the l_1 representation.

The non-linear realisation of $E_{11} \otimes_s l_1$ is constructed from a group element $g \in E_{11} \otimes_s l_1$ which can be written as

$$g = g_l g_E \quad (3.1)$$

where

$$g_E = e^{A^{a_1 \dots a_3} R_{a_1 \dots a_3}} e^{A^{a_1 \dots a_6} R_{a_1 \dots a_6}} e^{h^{a_1 \dots a_8, b} R_{a_1 \dots a_8, b}} \dots$$

$$e^{h_a{}^b K^a{}_b} \dots e^{h_{a_1 \dots a_8, b} R^{a_1 \dots a_8, b}} e^{A_{a_1 \dots a_6} R^{a_1 \dots a_6}} e^{A_{a_1 \dots a_3} R^{a_1 \dots a_3}} \quad (3.2)$$

and

$$g_l = e^{x^a P_a} e^{x_{ab} Z^{ab}} e^{x_{a_1 \dots a_5} Z^{a_1 \dots a_5}} \dots = e^{z^A L_A} \quad (3.3)$$

where we have denoted the generalised coordinates by z^A and the generators of the l_1 representation by L_A . Thus the non-linear realisation introduces a generalised space-time with the coordinates [8]

$$x^a, x_{ab}, x_{a_1 \dots a_5}, \dots \quad (3.4)$$

The fields that occur in the group element g_E are taken to depend on the generalised space-time that is the coordinates of equation (3.4). Since the l_1 representations contains all the brane charges [8,9,3,11] and this was responsible for the generalised space-time there is a one to one relation between the brane charges and the coordinates of the generalised space-time. Furthermore, for every field in E_{11} there corresponds an element in the l_1 representation [9]. As such for every field there is an associated coordinate in the generalised

space-time. For example, the metric corresponds to the point particle with charge P_a and coordinate x^a , the three form corresponds to the two brane with charge $Z^{a_1 a_2}$ and coordinate $x_{a_1 a_2}$, the six form corresponds to the five brane with charge $Z^{a_1 \dots a_5}$ and coordinate $x_{a_1 \dots a_5}$ and so on. As the discussion below makes clear, the non-linear realisation $E_{11} \otimes_s l_1$ automatically encodes a generalised geometry equipped with a generalised vielbein.

The non-linear realisation is by definition just a set of dynamics that is invariant under the transformations

$$g \rightarrow g_0 g, \quad g_0 \in E_{11} \otimes_s l_1, \quad \text{as well as} \quad g \rightarrow g h, \quad h \in I_c(E_{11}) \quad (3.5)$$

The group element g_0 is a rigid transformation, that is a constant, while h is a local transformation, that is it depends on the generalised space-time. As the generators in g_l form a representation of E_{11} the above transformations for $g_0 \in E_{11}$ can be written as

$$g_l \rightarrow g_0 g_l g_0^{-1}, \quad g_E \rightarrow g_0 g_E \quad \text{and} \quad g_E \rightarrow g_E h \quad (3.6)$$

As a consequence the coordinates are inert under the local transformations but transform under the rigid transformations as

$$z^A l_A \rightarrow g_0 z^A l_A g_0^{-1} = z^\Pi D(g_0^{-1})_\Pi{}^A L_A \quad (3.7)$$

Using the local transformation we may bring g_E into the form

$$g_E = e^{h_a{}^b K^a{}_b} \dots e^{h_{a_1 \dots a_8, b} R^{a_1 \dots a_8, b}} e^{A_{a_1 \dots a_6} R^{a_1 \dots a_6}} e^{A_{a_1 \dots a_3} R^{a_1 \dots a_3}} \quad (3.8)$$

Thus the theory contains the graviton field $h_a{}^b$ associated with the generators $K^a{}_b$ of $GL(11)$, as well as the gauge fields $A_{a_1 a_2 a_3}$ and its dual $A_{a_1 \dots a_6}$ associated with the level one and two generators $R^{a_1 a_2 a_3}$ and $R^{a_1 \dots a_6}$ respectively. Furthermore, in addition we have a field $h_{a_1 \dots a_8, b}$ corresponding to the generator $R^{a_1 \dots a_8, b}$ which is the dual field of gravity [1]. The parameterisation of the group element differs from that used in some earlier works on E_{11} , but this does not affect any physical results.

The dynamics is usually constructed from the Cartan forms $\mathcal{V} = g^{-1} dg$ as these are inert under the E_{11} rigid transformations of equation (3.5) and only transform under the local transformations as

$$\mathcal{V} \rightarrow h^{-1} \mathcal{V} h + h^{-1} dh \quad (3.9)$$

Hence if we use the Cartan forms, the problem of finding a set of field equations which are invariant under equation (3.5) reduces to finding a set that is invariant under the local subalgebra $I_c(E_{11})$, that is the transformations of equation (3.9).

The Cartan forms can be written as

$$\mathcal{V} = \mathcal{V}_E + \mathcal{V}_l \quad (3.10)$$

where

$$\mathcal{V}_E = g_E^{-1} dg_E \quad \text{and} \quad \mathcal{V}_l = g_E^{-1} (g_l^{-1} dg_l) g_E \quad (3.11)$$

The first part \mathcal{V}_E is just the Cartan form for E_{11} while \mathcal{V}_l is a sum of generators in the l_1 representation. While both \mathcal{V}_E and \mathcal{V}_l are invariant under rigid transformations and under local transformations they change as

$$\mathcal{V}_E \rightarrow h^{-1}\mathcal{V}_E h + h^{-1}dh \quad \text{and} \quad \mathcal{V}_l \rightarrow h^{-1}\mathcal{V}_l h \quad (3.12)$$

Let us evaluate the E_{11} part of the Cartan form

$$\mathcal{V}_E = dz^\Pi G_{\Pi,\star} R^\star = G_a{}^b K^a{}_b + G_{c_1\dots c_3} R^{c_1\dots c_3} + G_{c_1\dots c_6} R^{c_1\dots c_6} + G_{c_1\dots c_8,b} R^{c_1\dots c_8,b} + \dots \quad (3.13)$$

where \star denotes the indices on the generators of E_{11} . Explicitly one finds that

$$\begin{aligned} G_a{}^b &= (e^{-1}de)_a{}^b, \quad G_{c_1\dots c_3} = \tilde{D}A_{c_1\dots c_3}, \\ G_{c_1\dots c_6} &= \tilde{D}A_{c_1\dots c_6} - A_{[c_1\dots c_3}\tilde{D}A_{c_4\dots c_6]} \\ G_{c_1\dots c_8,b} &= \tilde{D}h_{c_1\dots c_8,b} - A_{[c_1\dots c_3}\tilde{D}A_{c_4c_5c_6}A_{c_7c_8]b} + 3A_{[c_1\dots c_6}\tilde{D}A_{c_7c_8]b} \\ &\quad + (A_{[c_1\dots c_3}\tilde{D}A_{c_4c_5c_6}A_{c_7c_8]b} - 3A_{[c_1\dots c_6}\tilde{D}A_{c_7c_8]b}) \end{aligned} \quad (3.14)$$

where $e_\mu{}^a \equiv (e^h)_\mu{}^a$ and

$$\tilde{D}A_{c_1\dots c_3} \equiv dA_{c_1c_2c_3} + ((e^{-1}de)_{c_1}{}^b A_{bc_2c_3} + \dots \quad (3.15)$$

where $+\dots$ denotes the action of $(e^{-1}de)$ on the other indices with analogous expressions for other quantities. In the last expression of equation (3.14) we have subtracted the totally anti-symmetric part corresponding to the fact that the generator obeys the condition $R^{[c_1\dots c_8,b]} = 0$. Evaluating this expression we find that

$$\begin{aligned} G_{c_1\dots c_8,b} &= \tilde{D}h_{c_1\dots c_8,b} - A_{[c_1\dots c_3}\tilde{D}A_{c_4c_5c_6}A_{c_7c_8]b} + 2A_{[c_1\dots c_6}\tilde{D}A_{c_7c_8]b} \\ &\quad + 2\tilde{D}A_{[c_1\dots c_5]b}A_{c_6c_7c_8}] \end{aligned} \quad (3.16)$$

We note that

$$A_{[c_1\dots c_3}\tilde{D}A_{c_4c_5c_6}A_{c_7c_8]b} = 0 \quad (3.17)$$

Let us now evaluate the part of the Cartan form in equation (3.10) containing the generators of the l_1 representation; we may write it as

$$\mathcal{V}_l = g^{-1}dg = dz^\Pi E_\Pi{}^A l_A = g_E^{-1}(dx^a P_a + dx_{ab}Z^{ab} + dx_{a_1\dots a_5}Z^{a_1\dots a_5} + \dots)g_E \quad (3.18)$$

Using equation (2.18-21) we find that $E_\Pi{}^A$, viewed as a matrix, is given at low orders by

$$E = (dete)^{-\frac{1}{2}} \begin{pmatrix} e_\mu{}^a & -3e_\mu{}^c A_{cb_1b_2} & 3e_\mu{}^c A_{cb_1\dots b_5} + \frac{3}{2}e_\mu{}^c A_{[b_1b_2b_3}A_{|c|b_4b_5]} \\ 0 & (e^{-1})_{[b_1}{}^{\mu_1}(e^{-1})_{b_2]}{}^{\mu_2} & -A_{[b_1b_2b_3}(e^{-1})_{b_4}{}^{\mu_1}(e^{-1})_{b_5]}{}^{\mu_2} \\ 0 & 0 & e^{-1})_{[b_1}{}^{\mu_1} \dots (e^{-1})_{b_5]}{}^{\mu_5} \end{pmatrix} \quad (3.19)$$

This illustrates the fact that the generalised space-time leads to a generalised tangent space, which in this case has the usual tangent space, two forms, five forms and higher objects. In general the tangent space can be read off from the l_1 representation in an obvious way. The l_1 representation appropriate to ten and d dimensions is found by decomposing E_{11} into $GL(d) \otimes E_{11-d}$ and the results can be found in [4,9,11,21]. The tangent space group is $I_c(E_{11})$; at lowest level this is $O(d) \otimes O(d)$ for the IIA theory in ten dimensions while in d dimensions it is $SO(d) \otimes I_c(E_{11-d})$

Our task is to find a set of dynamics which is invariant under the rigid and local transformations of equation (3.5) and with this in mind we now consider in more detail the transformations of the two parts of the Cartan form beginning with that of E_{11} part. As noted above the Cartan forms only transform under the local $I_c(E_{11})$ transformations. It is useful to introduce the operation $g^* = (I_c(g))^{-1}$ on the group. While I_c is an automorphism, i.e. on two group elements $I_c(g_1 g_2) = I_c(g_1) I_c(g_2)$, the action of $*$ reverses the order, that is $(g_1 g_2)^* = (g_2)^* (g_1)^*$. The action of $*$ on the algebra is given by $A^* = -I_c(A)$ and $(AB)^* = B^* A^*$. A group element belonging to $I_c(E_{11})$ obeys $h^* = h^{-1}$ and the two transformations of equation (3.5) imply that $g^* \rightarrow h^{-1} g^* (g_0)^*$. We write the Cartan forms \mathcal{V}_E as

$$\mathcal{V}_E = P + Q, \quad \text{where } P = \frac{1}{2}(\mathcal{V}_E + \mathcal{V}_E^*), \quad Q = \frac{1}{2}(\mathcal{V}_E - \mathcal{V}_E^*) \quad (3.20)$$

and then the transformations of equation (3.12) becomes

$$P \rightarrow h^{-1} P h, \quad Q \rightarrow h^{-1} Q h + h^{-1} d h \quad (3.21)$$

Examining equation (3.13) we find that

$$\begin{aligned} P = & \frac{1}{2} G_a{}^b (K^a{}_b + K_b{}^a) + \frac{1}{2} G_{c_1 \dots c_3} (R^{c_1 \dots c_3} + R_{c_1 \dots c_3}) + \frac{1}{2} G_{c_1 \dots c_6} (R^{c_1 \dots c_6} - R_{c_1 \dots c_6}) \\ & + \frac{1}{2} G_{c_1 \dots c_8, b} (R^{c_1 \dots c_8, b} + R^{c_1 \dots c_8, b}) + \dots \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} Q = & \frac{1}{2} G_a{}^b (K^a{}_b - K_b{}^a) + \frac{1}{2} G_{c_1 \dots c_3} (R^{c_1 \dots c_3} - R_{c_1 \dots c_3}) + \frac{1}{2} G_{c_1 \dots c_6} (R^{c_1 \dots c_6} + R_{c_1 \dots c_6}) \\ & + \frac{1}{2} G_{c_1 \dots c_8, b} (R^{c_1 \dots c_8, b} - R^{c_1 \dots c_8, b}) + \dots \end{aligned} \quad (3.23)$$

We note that the connection Q contains the same objects as the covariant quantity P .

Taking $h = 1 - \Lambda_{a_1 a_2 a_3} S^{a_1 a_2 a_3}$, the local transformations of P of equation (3.21) implies, using equations (2.12-14) and equations (2.18-21) that

$$\begin{aligned} \delta G^{ab} &= 18 \Lambda^{c_1 c_2 b} G_{c_1 c_2}{}^a - 2 \delta^{ab} \Lambda^{c_1 c_2 c_3} G_{c_1 c_2 c_3}, \\ \delta G_{a_1 a_2 a_3} &= -\frac{5!}{2} G_{b_1 b_2 b_3 a_1 a_2 a_3} \Lambda^{b_1 b_2 b_3} - 6 G^c{}_{[a_1} \Lambda_{|c| a_2 a_3]}, \end{aligned}$$

$$\begin{aligned}\delta G_{a_1 \dots a_6} &= 2\Lambda_{[a_1 a_2 a_3} G_{a_4 a_5 a_6]} - 8.7.2 G_{b_1 b_2 b_3 [a_1 \dots a_5, a_6]} \Lambda^{b_1 b_2 b_3} + 8.7.2 G_{b_1 b_2 [a_1 \dots a_5 a_6, b_3]} \Lambda^{b_1 b_2 b_3} \\ \delta G_{a_1 \dots a_8, b} &= -3G_{[a_1 \dots a_6} \Lambda_{a_7 a_8] b} + 3G_{[a_1 \dots a_6} \Lambda_{a_7 a_8 b]} \end{aligned} \quad (3.24)$$

Let us now turn our attention to the transformation of the part of the Cartan form in the direction of the l_1 representation, that is \mathcal{V}_l . At lowest level equation (3.12) implies that

$$E_\Pi^{A'} = E_\Pi^B D(h)_B^{A'}, \quad \text{and for the inverse} \quad (E^{-1})_A^{\Pi'} = D(h^{-1})_A^B (E^{-1})_B^{\Pi} \quad (3.25)$$

if we define $h^{-1} L_A h = D(h)_A^B L_B$. At lowest levels this implies the local transforms

$$\begin{aligned}\delta E_\Pi^a &= -6E_{\Pi b_1 b_2} \Lambda^{b_1 b_2 a}, \quad \delta E_{\Pi a_1 a_2} = 3\Lambda_{a_1 a_2 b} E_\Pi^b, \dots \\ \delta (E^{-1})_a^\Pi &= -3(E^{-1})^{b_1 b_2 \Pi} \Lambda_{b_1 b_2 a}, \quad \delta (E^{-1})^{a_1 a_2 \Pi} = 6\Lambda^{a_1 a_2 b} (E^{-1})_b^\Pi, \dots \end{aligned} \quad (3.26)$$

Even though the Cartan forms are invariant under the rigid transformations, E_Π^A and $G_{\Pi\star}$ are not as the transformation of z^Π of equation (3.7) implies a corresponding inverse transformation acting on the Π index of these two objects. Thus E_Π^A transforms under a local transformation on its A index and by the inverse of the coordinate transformation on its Π index. As such we can think of it as a generalised vielbein. We can rewrite the Cartan form of $E_{11} \otimes_s l_1$ as

$$\mathcal{V} = g^{-1} dg = dz^\Pi E_\Pi^A (L_A + G_{A,\star} R^*) \quad (3.27)$$

where $G_{A,\star} = (E^{-1})_A^\Pi G_{\Pi,\star}$. At low levels $(E^{-1})_A^\Pi$ is the inverse of the matrix of equation (3.19). Clearly $G_{A,\star}$ is inert under rigid transformations, but it transforms under local transformations as in equation (3.24) on its \star index and as the inverse generalised vielbein on its A index, that is as in equation (3.26).

Thus if we choose to construct the dynamics out of $G_{A,\star}$ we need only worry about the local transformations as the rigid transformations are automatically taken care of. Hence we seek a set of equations that are first order in $G_{A,\star}$ and invariant under $I_c(E_{11})$ transformations; thus we are left with a problem in group theory.

We will now focus our attention on finding the terms in such an invariant dynamics that involve the three and six form gauge fields and the coordinates x^a, x_{ab} and $x_{a_1 \dots a_5}$. We can solve this problem using a trick which may not generalise to the full system. In [8] it was shown at the lowest levels that $I_c(E_{11})$ is the group $SL(32)$ and the generators $P_a, Z^{ab}, Z^{a_1 \dots a_5}$ can be collected together in the matrix

$$Z_\alpha^\beta = (\gamma^a P_a + \frac{\gamma^{ab}}{2} Z^{ab} + \frac{\gamma^{a_1 \dots a_5}}{5!} Z^{a_1 \dots a_5})_\alpha^\beta \quad (3.28)$$

where $\alpha, \beta = 1, \dots, 32$ and the γ^a matrices are elements of the eleven dimensional Clifford algebra. In fact these first few components of the l_1 representation are the charges that occur in the eleven dimensional supersymmetry algebra and in the above equation we recognise the right hand side as the result of the anti-commutators of two supersymmetry generators; it is the most general symmetric matrix. One can verify that the local

transformations of these generators of the l_1 representations, given in equation (2.24), can be written as [8]

$$[S_{a_1 a_2 a_3}, Z_\alpha^\beta] = \left\{ \frac{\gamma_{a_1 a_2 a_3}}{2}, Z \right\}_\alpha^\beta \quad (3.29)$$

This labeling of the generators as a bispinor implies a corresponding labeling of the coordinates and so the generalised vielbein which we can define as

$$\mathcal{V}_s = dz^\Pi E_\Pi^A L_A \equiv dz^\Pi E_{\Pi\beta}^\alpha Z_\alpha^\beta = \frac{1}{32} Tr(E_\Pi Z) \quad (3.30)$$

Comparing with the expression of equation (3.18) we find that

$$E_{\Pi\beta}^\alpha = (\gamma_a E_\Pi^a - \gamma^{a_1 a_2} E_{\Pi a_1 a_2} + \gamma^{a_1 \dots a_5} E_{\Pi a_1 \dots a_5})_\beta^\alpha \quad (3.31)$$

we have used that

$$\frac{1}{32} Tr(\gamma^{a_1 \dots a_p} \gamma_{b_1 \dots b_p}) = (-1)^{\frac{p(p-1)}{2}} p! \delta_{b_1 \dots b_p}^{a_1 \dots a_p} \equiv (-1)^{\frac{p(p-1)}{2}} p! \delta_{[b_1}^{a_1} \dots \delta_{b_p]}^{a_p} \quad (3.33)$$

Using equations (3.12) and (3.29) we find that the infinitesimal transformation of the generalised vielbein under a local transformation when written in terms of the bispinor notation is given by

$$\delta E_{\Pi\beta}^\alpha = \frac{1}{2} \{ \gamma^{a_1 a_2 a_3} \Lambda_{a_1 a_2 a_3}, E_\Pi \}_\beta^\alpha \quad (3.34)$$

We can define the inverse generalised vielbein by $E_{\Pi\beta}^\alpha (E^{-1})_\alpha^{\beta\Lambda} = \delta_\Pi^\Lambda$ and it transforms under a local transformation as

$$\delta (E^{-1})_\alpha^{\beta\Lambda} = -\frac{1}{2} \{ \gamma^{a_1 a_2 a_3} \Lambda_{a_1 a_2 a_3}, (E^{-1})^\Lambda \}_\alpha^{\beta\Lambda} \quad (3.35)$$

Let us also reformulate the transformations of the E_{11} part of the Cartan form which is in the coset, that is the object P contained in equation (3.22), when restricted to contain only the three and six form fields in terms of gamma matrices. Let us define

$$\mathcal{P}_{\Pi,\alpha}^\beta = \left(\frac{\gamma^{a_1 a_2 a_3}}{2} G_{\Pi, a_1 a_2 a_3} + \frac{\gamma^{a_1 \dots a_6}}{4} G_{\Pi, a_1 \dots a_6} \right)_\alpha^\beta \quad (3.36)$$

One can then verify that the transformation

$$\delta \mathcal{P}_{\Pi,\alpha}^\beta = \frac{1}{2} [\gamma^{a_1 a_2 a_3} \Lambda_{a_1 a_2 a_3}, \mathcal{P}_\Pi]_\alpha^\beta \quad (3.37)$$

leads to the transformations of equation (3.24) for the parts of the Cartan forms corresponding to the three form and six form fields.

As discussed above we can convert the world index on the Cartan form into a tangent space index using the inverse generalised vielbein; using the bispinor notation we define

$$\mathcal{P}_{\alpha^\beta, \gamma}^\delta \equiv (E^{-1})_\alpha^{\beta\Pi} \mathcal{P}_{\Pi, \gamma}^\delta \quad (3.38)$$

Let us define

$$\mathcal{P}_\alpha{}^\beta \equiv \mathcal{P}_\alpha{}^\delta{}_{,\delta}{}^\beta \quad (3.39)$$

which, using (3.35) and (3.37), we find to transform as

$$\delta \mathcal{P}_\alpha{}^\beta = -\frac{1}{2} \{ \gamma^{a_1 a_2 a_3} \Lambda_{a_1 a_2 a_3}, \mathcal{P} \}_\alpha{}^\beta \quad (3.40)$$

Thus we have found an object which is inert under the rigid transformations and transforms covariantly under the local transformation and as such we have found a candidate for the equation of motion. In fact we have two possible covariant objects as we can symmetrise and anti-symmetrise on the α and β indices after lowering the latter with the inverse charge conjugation matrix. We note that the eleven dimensional gamma matrices $\gamma^a C^{-1}$, $\gamma^{a_1 a_2} C^{-1}$ and $\gamma^{a_1 \dots a_5} C^{-1}$ are symmetric while C^{-1} , $\gamma^{a_1 a_2 a_3} C^{-1}$ and $\gamma^{a_1 \dots a_4} C^{-1}$ are anti-symmetric. Let us consider the anti-symmetric part which we can set to zero to obtain the invariant equation

$$\begin{aligned} 0 &= \frac{1}{2} (\mathcal{P}_\alpha{}^\delta (C^{-1})_{\delta\gamma} - \mathcal{P}_\gamma{}^\delta (C^{-1})_{\delta\alpha}) \\ &= (\gamma^{a_1 a_2 a_3 a_4} C^{-1})_{\alpha\gamma} \mathcal{P}_{a_1 a_2 a_3 a_4} + (\gamma^{a_1 a_2 a_3} C^{-1})_{\alpha\gamma} \mathcal{P}_{a_1 a_2 a_3} + (C^{-1})_{\alpha\gamma} \mathcal{P} \end{aligned} \quad (3.41)$$

Thus we find the equations

$$\begin{aligned} 2\mathcal{P}_{a_1 a_2 a_3 a_4} &\equiv G_{[a_1, a_2 a_3 a_4]} - \frac{3.5}{2} G_{b_1 b_2,}{}^{b_1 b_2}{}_{a_1 a_2 a_3 a_4} - \frac{1}{2.4!} \epsilon_{a_1 a_2 a_3 a_4}{}^{b_1 \dots b_7} G_{b_1, b_2 \dots b_7} \\ &\quad - \frac{1}{2} G_{b_1 b_2 [a_1 a_2 a_3,}{}^{b_1 b_2}{}_{a_4]} + \frac{5}{4.4!} \epsilon_{a_1 a_2 a_3 a_4}{}^{b_1 \dots b_7} G_{c_1 c_2 b_1 b_2 b_3,}{}^{c_1 c_2}{}_{b_4 \dots b_7} = 0 \end{aligned} \quad (3.42)$$

$$\begin{aligned} 2\mathcal{P}_{a_1 a_2 a_3} &\equiv -6G_{[a_1 | b,}{}^b{}_{a_2 a_3]} + \frac{1}{4} \epsilon_{a_1 a_2 a_3}{}^{b_1 \dots b_8} G_{b_1 b_2. b_3 \dots b_8} \\ &\quad + \frac{1}{3!.5!} \epsilon_{a_1 a_2 a_3}{}^{b_1 \dots b_8} G_{b_1 \dots b_5, b_6 \dots b_8} + \frac{5.3}{2} G^{c_1 c_2 c_3 c_4}{}_{[a_1, | c_1 c_2 c_3 c_4 | a_2 a_3]} = 0 \end{aligned} \quad (3.43)$$

and

$$2\mathcal{P} \equiv \frac{1}{2.5!.11!} \epsilon^{a_1 \dots a_{11}} G_{a_1 \dots a_5, a_6 \dots a_{11}} = 0 \quad (3.44)$$

In finding these equations use was made of the identity

$$\gamma^{a_1 \dots a_p} \gamma_{b_1 \dots b_q} = \sum_r (-1)^{\frac{r(r-1)}{2}} (-1)^{(p-r)r} \frac{p!q!}{r!(p-r)!(q-r)!} \delta_{[b_1 \dots b_r}^{[a_1 \dots a_r} \gamma^{a_{r+1} \dots a_p]}{}_{b_{r+1} \dots b_q]} \quad (3.45)$$

and equation (3.31). In considering these equations it is important to recall that we have set to zero all contributions involving the gravity and dual gravity fields.

We note that only the first of these equations involves the derivative with respect to the usual coordinates x^a of space-time. At the linearised level this equation is given by

$$\partial_{[a_1} A_{a_2 a_3 a_4]} - \frac{3.5}{2} \partial_{b_1 b_2} A^{b_1 b_2}{}_{a_1 a_2 a_3 a_4} - \frac{1}{2.4!} \epsilon_{a_1 a_2 a_3 a_4}{}^{b_1 \dots b_7} \partial_{[b_1} A_{b_2 \dots b_7]}$$

$$-\frac{1}{2}\partial_{b_1 b_2 [a_1 a_2 a_3} A^{b_1 b_2}{}_{a_4]} + \frac{5}{4 \cdot 4!} \epsilon_{a_1 a_2 a_3 a_4}{}^{b_1 \dots b_7} \partial_{c_1 c_2 b_1 b_2 b_3} A^{c_1 c_2}{}_{b_4 \dots b_7} = 0 \quad (3.46)$$

where $\partial_a = \frac{\partial}{\partial x^a}$, $\partial^{ab} = \frac{\partial}{\partial x_{ab}}$ and $\partial^{a_1 \dots a_5} = \frac{\partial}{\partial x_{a_1 \dots a_5}}$.

If we were to restrict the dependence of the fields to only be on x^a then the last equation would be the correct equation of motion for the three and six form fields at linearised level, Indeed at the full non-linear level we find the field equation

$$F_{c_1 \dots c_4} = \frac{1}{7.6.2} \epsilon_{c_1 \dots c_4}{}^{b_1 \dots b_7} F_{b_1 \dots b_7} \quad (3.47)$$

where

$$F_{c_1 \dots c_4} \equiv 4\partial_{[c_1} A_{c_2 \dots c_4]} \quad (3.48)$$

and

$$F_{c_1 \dots c_7} \equiv 7(\partial_{[c_1} A_{c_2 \dots c_7]} - A_{[c_1 c_2 c_3} \partial_{c_4} A_{c_5 c_6 c_7]}) \quad (3.49)$$

The reader may wish to explicitly vary equation (3.42) using equations (3.24) and (3.26) and show that the resulting terms which contain the usual space-time derivative ∂_a do actually cancel. A useful intermediate result is that

$$\delta(G_{[a_1, a_2 a_3 a_4]} - \frac{3.5}{2} G_{b_1 b_2,}{}^{b_1 b_2}{}_{a_1, a_2 a_3 a_4}) = +\frac{5!.7}{8} G_{[b_1, b_2 b_3 a_1 \dots a_4]} \lambda^{b_1 b_2 b_3} \quad (3.50)$$

from which we see that although the left-hand side is not a field strength both terms conspire so as to give a variation that is a field strength, as the equation of motion requires.

The most general invariant equation linear in generalised space-time derivatives would be a sum of the symmetric and antisymmetric parts of $\mathcal{P}_\gamma{}^\delta (C^{-1})_{\delta\alpha}$ with arbitrary coefficients. However, the symmetric part involves terms such as $G_{b, a_1 a_2 a_3} \eta^{b a_1}$ which are clearly not gauge invariant if one restricts the dependence on the generalised space-time to be only on x^a . The strategy used here is similar to that used in the original paper [14] on gravity, except that they used conformal symmetry to fix the constants, and the approach used in the early E_{11} papers, such as [1,12]. However, an extension of this procedure was applied in [15,16] where the non-linear realisation of $E_{11} \otimes_s l_1$, appropriate to four dimensions and at lowest level, was carried out. This meant keeping only the coordinates of the four dimensional space-time and those in the 56 dimensional representation of E_7 from the l_1 representations. An invariant action was then found that contained a number of undetermined constants. The constants were then fixed by demanding general coordinate invariance once the fields had been restricted to depend on the coordinates of the four dimensional space-time and only the usual seven of the 56 other coordinates. It is this strategy we are following here.

4. Discussion

The variation of equation (3.42) under the transformations of equation (3.24) will lead to a duality relation between the derivative of the graviton and that of the dual graviton. It would certainly be interesting to find what these equations are and if they really do describe the correct dynamics for gravity as it appears in the framework of eleven

dimensional supergravity once we neglect the higher E_{11} fields and the higher coordinates. Should this be the case then the E_{11} conjecture [1,8] will be shown to be true, namely that the non-linear realisation of $E_{11} \otimes_s l_1$ is an extension of the dynamics of eleven dimensional supergravity. We hope to report on this soon. Although when E_{11} was first proposed the meaning of the higher fields was unknown, we have now come to understand the physical significance for large numbers of the higher level fields. The result in this paper suggests that the additional coordinates will also have a physical meaning. It would be very straightforward to extend the results in this paper to the IIA and IIB theories in ten dimensions and the theories in d dimensions using the techniques previously employed [1,2,3,4,5,6,7].

The non-linear realisation of $E_{11} \otimes_s l_1$ has not been systematically computed before. In the early papers on E_{11} only the coordinate x^a was used and the local subalgebra was taken to be just the Lorentz algebra. As a result much of the power of the non-linear realisation was lost. Nonetheless many of the features of the supergravity theories were recovered, for example the fields strengths for all gauged supergravity theories in five dimensions [5]. This paper should open the way to the systematic computation of the $E_{11} \otimes_s l_1$ non-linear realisation and so the dynamics it contains.

Above we simply deleted the dependence of the fields on the higher coordinates. However, it remains to understand what physical procedure one should use to reduce the dependence on the fields on the generalised space-time. The work of reference [17] suggested that even though the full theory was $E_{11} \otimes_s l_1$ invariant only part of the l_1 representation occurred in the second quantised field theory. In particular although the first quantised theory involved all of the coordinates of the l_1 representation, the choice of representation of the commutators that takes one from the first to the second quantised theory required one to choose only part of the l_1 representation. However, one can make different choices of which part of the l_1 representation one takes and these should be equivalent and related by E_{11} transformations. It would be interesting to really understand how this works. However, it is likely that a simple truncation will not be the only allowed possibility; indeed in the construction of all the five dimensional supergravity theories [5] we found a much more subtle procedure involving a slice that included part of E_{11} .

To close it could be helpful to discuss the relationship between the various works on generalised geometry. This paper is based on the 2003 proposal to consider the non-linear realisation of $E_{11} \otimes_s l_1$ [8], however, there are several other approaches. A generalised space-time appeared in the context of string dynamics where the usual space-time was extended to include an additional coordinate y_a . This was done in such a way as to encode the (first quantised) dynamics of the string in an $O(D,D)$ symmetric manner [18,19]; a generalisation to the membrane was also given [20]. In fact the dynamics of strings and membranes can be formulated as a non-linear realisation of $E_{11} \otimes_s l_1$ [21]. The difference from the non-linear realisation studied above is that a different choice of local subalgebra is taken and the coordinates associated with the l_1 representation become fields. The non-linear realisation can be carried out so as to include the background supergravity fields that belong to the E_{11} part of the non-linear realisation in the same way as above. If one takes the non-linear realisation of $E_{11} \otimes_s l_1$ appropriate to ten dimensions and at lowest level one finds the generalised space-time and the string dynamics given in [18,19,20]. Carrying

out this non-linear realisation in d dimensions at lowest order one finds the coordinates of d dimensional Minkowski space-time and in addition scalar coordinates belong to the 10, 16, 27, 56 and $248 \oplus 1$ of $SL(5)$, $SO(5,5)$, E_6 , E_7 and E_8 for d equal to seven, six, five and four and three dimensions respectively [3,11,21,5]. These are the same coordinates as arises at level zero in the non-linear realisation of $E_{11} \otimes_s l_1$ used to find the supergravity theories in d dimensions in the absence of strings and branes.

Another version of generalised geometry was inspired by a version of closed string field theory [22] and the papers of [23]. It goes by the name of doubled field theory as it also doubles the number of coordinates to have a x^a and y_a in order to encode an $O(10,10)$ symmetry [24]. The field theory is defined on this space contains the same fields as in the NS-NS sector of the ten dimensional superstring. An action was constructed and if one restricts the fields to depend on just the usual space-time, that is just on x^a , one finds the well known action for the NS-NS sector [33,34]. However, doubled field theory is just a sub case of the non-linear realisation of $E_{11} \otimes_s l_1$. To be precise it is the non-linear realisation of $E_{11} \otimes_s l_1$ at lowest level in the decomposition appropriate to the IIA theory [25]. This is a very straightforward systematic calculation that requires no guess work and took only six pages in [25] to present in all detail. The advantage of viewing this as a non-linear realisation of $E_{11} \otimes_s l_1$ is that it places the construction in a wider conceptual framework in which the true nature of the symmetries is apparent. For example, the presence of the $GL(1)$ symmetry in addition to $O(10,10)$ becomes clear and one can construct the extension to the next level [26]. This is also very straightforward and one finds [26] the R-R part of the well known supergravity equations of motion. One could also compute even higher levels involving fields beyond that of the usual maximal supergravity theories and so find new physics.

The approach of [19,20] just mentioned above had the aim of encoding some duality symmetries in the first quantised dynamics. This work was taken up in [27] which derived a generalised metric from the first quantised theory and used this to construct invariant $SL(5)$ dynamics for fields living on a space whose coordinates belonged to the ten dimensional representation of $SL(5)$. This work was then generalised to the duality group $SO(5,5)$ [28]. This was in agreement with the general framework of [21] and the non-linear realisation of $E_{11} \otimes_s l_1$ taking into account that the coordinates of the l_1 representations in d dimensions at level zero are those mentioned just above. Very recently it was shown [29] in detail how these theories [27,28] were the non-linear realisation of $E_{11} \otimes_s l_1$ appropriate to seven and six dimensions at lowest level. The work of [29] also contained the generalisation to find the analogues of these results in five and four dimensions and so involving the duality groups E_6 and E_7 respectively. The precise relationship to the work of [15,16] which earlier computed the non-linear realisation of $E_{11} \otimes_s l_1$ at lowest level in four dimensions and used the generalised space consisting of the usual coordinates of space-time and the coordinates in the 56 dimensional representation of E_7 has yet to be clarified.

There is yet another approach inspired by the work of Hitchin [30] and Gualtieri [31]. This introduced an extended tangent space, associated with $O(D,D)$ but does not extend our usual notion of space-time, see for example [32] and references therein. There has not been a study to investigate the connection to the non-linear realisation of $E_{11} \otimes_s l_1$. However, the generalised tangent space in ten dimensions which encodes $O(D,D) \otimes GL(1)$

with tangent group $O(10) \otimes O(10)$ [32], is precisely the same as that which arises in the non-linear realisation of $E_{11} \otimes_s l_1$ appropriate to ten dimensions at lowest level [25,26]. As we mentioned below equation (3.19) the generalised tangent space is in general just that given by the l_1 representation and the tangent space group is $I_c(E_{11})$. In eleven dimensions this is just the usual tangent space, the space of two forms and five forms and higher objects [8]. While in d dimensions we would find the usual tangent space, scalars belong to the 10, $\bar{16}$, $\bar{27}$, 56 and $248 \oplus 1$ of $SL(5)$, $SO(5,5)$, E_6 , E_7 and E_8 for d equal to seven, six, five, four and three dimensions respectively as well as higher level objects [3,21,5,11]. This leads one to suspect that if one carries out the $E_{11} \otimes_s l_1$ non-linear realisation, but at the end sets all the fields to depend on just the usual coordinates x^a then one might obtain this approach.

Acknowledgments

I wish to thank Dario Martelli for discussions and STFC for support from the grant given to the theoretical physics group in the Mathematics Department at King's.

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N. Note added on the equation of motion in the gravity sector

In this note added we will derive, from the $E_{11} \otimes_s l_1$ non-linear realisation, the equation of motion that relates the usual field of gravity to its dual field. We will work only to the level that the dual graviton occurs and so we will not derive the terms involving the higher level E_{11} fields associated with gravity, nor shall we derive the terms that contain generalised space-time derivatives for coordinates beyond the lowest level, that is, we will only find terms with derivatives with respect to the coordinates of our customary space-time.

The notion of a dual graviton was first introduced by Curtwright [N1]. This work was based on an equation that involved the Riemann tensor. The dual graviton $h_{a_1 \dots a_8, b}$ arises automatically as a field at level three in the E_{11} non-linear realisation and it was proposed that it was related to the vielbein by a duality relation that is first order in derivatives [1]. In reference [1] a non-linear equation, involving the vielbein and a field $Y_{b_1 \dots b_9, b}$, which did correctly described full gravity was given and it was also shown that at the linearised level could one substitute the field $Y_{b_1 \dots b_9, b}$ for the derivative of the dual gravity field and find the correct equation for linearised gravity. This left the situation for the full non-linear theory unresolved. However, in reference [13] an equation involving the vielbein, its dual and the field $Y_{b_1 \dots b_9, b}$, which was first order in derivatives, was given and shown the correctly describe gravity at all orders. This result seems to have gone somewhat unnoticed in the subsequent literature.

Although these results were encouraging, there remained the problem of how a duality relation between the vielbein and the dual gravity field would arise in E_{11} . This was discussed in reference [N2] and in [N3]; the latter includes considerations based on supersymmetry, but no firm contact with E_{11} was found.

Equation (3.42), which relates the three form gauge field to its dual six form field, was derived by introducing objects with spinor indices in a maneuver that may not generalise to the other fields. As such we begin this section by giving a more conventional derivation of this equation of motion. In particular we will explicitly vary equation (3.42) under the local $I_c(E_{11})$ symmetry variations of equations (3.24) and (3.26). We recall that the way the dynamical equations are constructed ensures that they are automatically invariant under the rigid $E_{11} \otimes_s l_1$ transformations. The equation is built from the Cartan forms of equation (3.14-16) whose indices are of the form $G_{\diamond, \bullet}$. The \bullet represent the indices that are contracted with the E_{11} generators that occur in the Cartan form and their variations under $I_c(E_{11})$ are given in equation (3.24). The \diamond indices arise from the forms, when converted to tangent indices with the inverse generalised vielbein, that occur in the Cartan form. These latter indices transform under the local transformations given in the second line of equation (3.26). To give an example, $G_{a_1 a_2, b_1 b_2 b_3}$ occurs with the E_{11} generators $R^{b_1 b_2 b_3}$ and the two form $dx^\pi E_\pi^{a_1 a_2}$ in the Cartan form. For simplicity we will often denote a generic Cartan form by just indicating the number of indices, for example we might denote the object just discussed by $G_{2,3}$. We note that this Cartan form contains generalised space-time derivatives that are with respect to $x^{a_1 a_2}$ and higher level coordinates, that is it does not contain derivatives with respect to the coordinates x^a of the usual space-time. Similar statements apply to all the other Cartan forms. Of course in principle one should write the variations of the \bullet and \diamond indices in one equation, but it can be convenient to

carry them out separately.

We now carry out the variation of the equation (3.42) under the local $I_c(E_{11})$ transformations for both of the two types of variation just mentioned, but keeping terms in the variation that contain Cartan forms $G_{\diamond, \bullet}$ with a \diamond index that is an a or an $a_1 a_2$, that is, we discard Cartan forms associated with generalised space-time derivatives with respect to $x_{a_1 \dots a_5}$ and higher level coordinates. We will first focus on terms in the variation that contain the three and six form gauge fields and later collect the terms that contain the gravity field and its dual. The terms in the variation that contain $G_{2,3} \Lambda_3$ are given by

$$+9G_{[a_1 a_2, \quad b_1 b_2 \quad a_3 \Lambda_{a_4] b_1 b_2} + 9G^{b_1 b_2}_{\quad, [a_1 a_2 | b_1 \Lambda_{b_2 | a_3 a_4]} - 18G_{[a_1 | b, \quad bc \quad a_2 \Lambda_{| c | a_3 a_4]} \quad (N.1)$$

In deriving this result we have already implemented the cancellations that occur between certain terms. While the terms involving $G_{2,6} \Lambda_3$ are given by

$$+\frac{1}{4.4}\epsilon_{a_1 a_2 a_3 a_4} \quad b_1 \dots b_7 \{ \Lambda_{b_1 c_1 c_2} G^{c_1 c_2}_{\quad, b_2 \dots b_7} + 15 \Lambda_{b_1 c_1 c_2} G_{b_2 b_3} \quad c_1 c_2 \quad, b_4 \dots b_7 + 5.6 \Lambda_{c_2 b_1 b_2} G_{b_3 c_1, \quad c_1 c_2 \quad b_4 \dots b_7} \} \quad (N.2)$$

where again we have not shown terms that cancel against each other.

The second and third terms of equation (N.1) can be written as

$$-9G_{[a_1 | b, \quad b \quad | c a_2]} \Lambda^c_{\quad a_3 a_4} \quad (N.3)$$

where the anti-symmetry on a_1, a_2, a_3 and a_4 in this particular equation is not indicated explicitly but is to be understood to be present.

The first and second terms in equation (N.2) can be written as

$$+\frac{7}{4}\epsilon_{a_1 a_2 a_3 a_4} \quad b_1 \dots b_7 \{ \Lambda_{b_1 \quad c_1 c_2} G_{[c_1 c_2, b_2 \dots b_7]} - \frac{6.4}{8.7} \Lambda_{b_1 c_1 c_2} G_{b_2 c_1, b_3 \dots b_7} \} \quad (N.4)$$

To lowest order equation (3.43) can be expressed as

$$G_{[b_1 b_2, b_3 \dots b_8]} = -\frac{6.4}{8!.3!} \epsilon_{b_1 \dots b_8} \quad e_1 e_2 e_3 G_{e_1 b, \quad e_2 e_3} \quad (N.5)$$

and substituting this into the first term in equation (N.4) we find that it cancels the terms two and three in equation (N.1).

The net effect of all this is that the local $I_c(E_{11})$ variation of equation (3.42) which contain only terms that involve the three and six form gauge fields, that is the Cartan forms $G_{2,3}$ and $G_{2,6}$, is given by

$$+9G_{[a_1 a_2, \quad b_1 b_2 \quad a_3 \Lambda_{a_4] b_1 b_2} + \frac{6}{4.4} \epsilon_{a_1 a_2 a_3 a_4} \quad b_1 \dots b_7 \{ -2 \Lambda_{b_1 c_1 c_2} G_{b_2 c_1, b_3 \dots b_7} + 5 \Lambda_{b_1 b_2 d} G_{b_3 c, \quad cd \quad b_4 \dots b_7} \} \quad (N.6)$$

These are the first term in equation (N.1), the second term in equation (N.4) and the third term in equation (N.2). These terms can be canceled by introducing gravity and dual gravity terms into equation (3.42) whose Cartan forms have the local variations of

equations (3.24) and (3.26). Carrying out these modifications equation (3.42) is now given by

$$\begin{aligned}
0 = & G_{[a_1, a_2 a_3 a_4]} - \frac{3.5}{2} G_{b_1 b_2, \quad b_1 b_2} a_1 a_2 a_3 a_4 - \frac{1}{2.4!} \epsilon_{a_1 a_2 a_3 a_4}^{b_1 \dots b_7} G_{b_1, b_2 \dots b_7} \\
& - \frac{1}{2} G_{b_1 b_2 [a_1 a_2 a_3, \quad b_1 b_2} a_4] + \frac{5}{4.4!} \epsilon_{a_1 a_2 a_3 a_4}^{b_1 \dots b_7} G_{c_1 c_2 b_1 b_2 b_3, \quad c_1 c_2} b_4 \dots b_7 \\
& - \frac{1}{2} G_{[a_1 a_2, a_3 a_4]} - \frac{7}{6} \epsilon_{a_1 a_2 a_3 a_4}^{b_1 \dots b_7} G_{c b_1, \quad c} b_2 \dots b_7 d, \quad d
\end{aligned} \tag{N.7}$$

The variation of the second to last and last terms in this equation cancel the first and second variations of equation (N.6) respectively. Thus we have found an equation for the three form and six form gauge fields that is invariant under the local $I_c(E_{11})$ transformations and so all the transformations of the non-linear realisation if we discard in the variation terms containing gravity and its dual and generalised space-time derivatives with respect to coordinates beyond the two form.

To better understand the above calculation it is useful to represent it in a schematic diagram, which is given in equation (N.8). We searching for an equation with four anti-symmetric indices constructed from the Cartan forms $G_{\diamond, \bullet}$ and the epsilon symbol, denoted by \star . In the diagram below we list all possible terms in a grid going with increasing level of the fields to the right and increasing level in the generalised space-time derivatives as one goes down. Where there is no term indicated it means that there is no such term that one can write down with the correct indices.

$$\begin{array}{ccccccc}
& & \leftarrow & G_{1,3} & \leftrightarrow & \star G_{1,6} & \rightarrow \\
& \uparrow & & \downarrow & & \updownarrow & \uparrow \\
G_{2,1,1} & \rightarrow & & & \leftarrow & G_{2,6} & \leftrightarrow \star G_{2,8,1} \\
& \updownarrow & & \uparrow & & \updownarrow & \updownarrow \\
\star G_{5,1,1} & \leftrightarrow & G_{5,3} & \leftrightarrow & \star G_{5,6} & \leftrightarrow & G_{5,8,1}
\end{array} \tag{N.8}$$

The arrows indicate what happens when one varies the terms under the local transformations. In particular, the vertical arrows indicate the effect of varying the first indices, that is the \diamond indices and the horizontal arrows the effect of varying the second indices, that is the \bullet indices. The terms so obtained contain the Cartan field at the site the arrow point to times the parameter Λ_3 . For example, the variation of the first index on $G_{1,3}$ leads to a term $\Lambda_3 G_{2,3}$ whose only other sources are given by following the arrows pointing to this site; for example, one such term arises from the variation of the second index of $G_{2,6}$. We note that just because a Cartan form is absent in the four index equation of motion this does not mean that the same Cartan form does not arise in the local variation of this equation as the variation has in general a different index structure.

Finally, we now compute the local variation of equation (3.42) keeping the remaining terms, that is, those that contain the gravity field or its dual. We will only keep derivatives with respect to the usual coordinates of space-time, that is, those with respect to x^a . We note that the resulting equation will have a different index structure to that for the three form field and as a result the spaces in the diagram where no such contribution can exist are different. Varying equation (N.7) we are interested in the terms correspond to the spaces in the diagram of equation (N.8) that are in the top line at the extreme left and

right hand ends and these can only come from the variation of $G_{a,b,c}$ ($G_{2,1,1}$), $G_{a,b_1\dots b_8,c}$ ($G_{2,8,1}$), by varying their first index, or $G_{1,3}$ and $G_{1,6}$ by varying their second index. Since there are no terms in the four index equation that involve the Cartan forms $G_{a,b,c}$ and $G_{a,b_1\dots b_8,c}$ the terms in the variation must either cancel or result in a new equation. In fact this new equation has three indices. The most general terms one can write down with three indices, with no particular symmetry, are given in the diagram of equation (N.9) whose interpretation is analogous to that for the diagram of equation (N.8).

$$\begin{array}{ccccccc}
G_{1,1,1} & \rightarrow & & & \leftarrow & \star G_{1,8,1} & \\
\downarrow & & \uparrow & & \uparrow & \downarrow & \\
& \rightarrow & G_{2,3} & \leftarrow & \star G_{2,6} & \leftrightarrow & \\
\uparrow & & \downarrow & & \downarrow & \uparrow & \\
G_{5,1,1} & \leftrightarrow & \star G_{5,3} & \leftrightarrow & G_{5,6} & \leftrightarrow & \star G_{5,8,1}
\end{array} \tag{N.9}$$

Setting the variation to zero we find that

$$3X_{[a_1 a_2, |c|] \Lambda^c_{a_3 a_4]} + 7\epsilon_{a_1 a_2 a_3 a_4}{}^{b_1 \dots b_7} \{G_{b_1, c_1 c_2 c_3 b_2 \dots b_6, b_7} \Lambda^{c_1 c_2 c_3} - G_{c_1, c_2 b_2 \dots b_7 d,}{}^d \Lambda^{c_1 c_2 b_1}\} = 0 \tag{N.10}$$

where

$$X_{a_1 a_2, c} = -G_{[a_1, a_2]c} - G_{[a_1, |c| a_2]} + G_{c, [a_1 a_2]} \tag{N.11}$$

We recall that $G_{c,a}{}^b = e_c{}^\mu (e \partial_\mu e)_a{}^b$ whereupon we recognise that $X_{ab,c} = w_{c,ab}$ where $w_{c,a}{}^b$ is the usual spin connection. Extracting the parameter in equation (N.10) we find the equation

$$3X_{[a_1 a_2, [c_1 \delta_{a_3 a_4}]^{c_2 c_3}] + 7\epsilon_{a_1 a_2 a_3 a_4}{}^{b_1 \dots b_7} \{G_{b_1, c_1 c_2 c_3 b_2 \dots b_6, b_7} - \delta_{b_1, [c_3} G_{c_1, c_2] b_2 \dots b_7 d,}{}^d\} = 0 \tag{N.12}$$

Setting $a_3 = c_2$ and $a_4 = c_3$ and summing over these indices we find that

$$\frac{7.2}{3} w_{c_1, a_1 a_2} - \frac{8}{3} w_{d[a_1,}{}^d \delta_{a_2]}^{c_1} + \epsilon_{a_1 a_2}{}^{b_1 \dots b_9} G_{b_1, b_2 \dots b_9, c_1} + \frac{7}{3} \epsilon_{a_1 a_2}{}^{c_1 b_1 \dots b_8} G_{b_1, b_2 \dots b_8 d,}{}^d = 0 \tag{N.13}$$

where we have substituted the spin connection.

We see that the equation has many of the correct features and in particular it relates the derivative of the vielbein, specifically the spin connection, to the derivative of the dual graviton field. Of course the full equation will contain higher level E_{11} fields and also derivatives with respect to the higher level coordinates of the generalised space-time. We observe that equation (N.13) contains not only the P part of the Cartan forms of equation (3.22) but also those of equation (3.23), that is, the Q part, which transform inhomogeneously under the local symmetry. In particular it contains the Cartan form $G_{c, [ab]}$ which transforms inhomogeneously under the local Lorentz group, see equation (3.21). The appearance of this Cartan form in the equations of motion is related to the fact that we have not chosen the local Lorentz transformation to set the anti-symmetric part of the graviton to zero, or equivalently, one of the off diagonal parts of the vielbein to zero. It follows that equation (N.13) does not strictly speaking hold as an equality as while the right-hand side transforms covariantly under local Lorentz transformations the left-hand

side transforms like the spin connection and so has an inhomogeneous term of the form $\partial^{c_1} \Lambda_{a_1 a_2}$ in its local variation. The E_{11} algebra implies that the dual gravity field satisfies the constraint $h_{[a_1 \dots a_8, b]} = 0$, but it is easy to see that the effect of allowing equality up to the above term is equivalent to relaxing this constraint and introducing a nine form object into the theory [13].

To find an invariant equation one must form the Riemann tensor $R_{\mu\nu, a}{}^b$ from the spin connection in the usual way. The corresponding terms involving the dual graviton do not vanish. However they do vanish at least at the linearised level if one then forms the usual contraction to form the Ricci tensor, by setting $\nu = b$ and then summing. Thus at the linearised level one would have the correct equation for gravity if it were it not for the following observation. We note that if we set c_1 and a_1 and then trace on the resulting index in equation (N.12) then we find that $w_{da,}{}^d = \partial_\mu (\det e_a{}^\mu) = 0$. As has just been remarked equation (N.13) only holds up to an inhomogeneous local Lorentz transformations, but unfortunately $w_{da_1,}{}^d$ is not of this form in general. Thus although equation (N.13) has many of the correct features it does not as it stands describe the correct equation for gravity. One might wonder if there is any term involving the dual gravity field that one could add and that would prevent $w_{da_1,}{}^d$ being zero. However, such a term would have to be constructed from the epsilon symbol, contain one derivative acting on the field $h_{a_1 \dots a_8, b}$ and have, after the contraction, only one index; one can easily see that this is not possible.

There are several ways out of this dilemma. The above computation could be wrong in its details and in particular if the magnitude of the second term in equation (N.13) was instead given by $-\frac{7}{15}$ then the terms involving the usual gravity field would also be traceless.

Another possibility is that there is some non-trivial dependence on the extra coordinates that leads to a non-trivial trace. However, the most likely possibility is that the trace can be non-zero if one includes the contribution of the higher level E_{11} fields. The gravity fields of the $E_{11} \otimes_s l_1$ non-linear realisation occur at level 0, 3, 6, 9, \dots and there is in fact no reason to believe that the full gravity equation arises only at the levels zero and three computed above. In fact at level six we find the fields

$$h_{10,6,2}, h_{9,8,1}, h_{11,4,3}, h_{11,5,1,1}, h_{10,7,1}, h_{11,6,1}, h_{10,8}, h_{11,7} \quad (N.14)$$

To contribute to the gravity equation we need an object with three indices once we have differentiated with respect to the usual space-time. As this leads to an odd number of indices we cannot use a single epsilon. It is very easy to find possible terms that have a trace, for example

$$\partial_a h^{b_1 \dots b_8 b}{}_{, b_1 \dots b_8, c} \quad (N.14)$$

The presence of this term would imply that the duality relation included also higher level Cartan forms on its right-hand side, however, in view of the non-linear nature of gravity this would not be unnatural.

This is the first time that the equation involving the gravity fields has been systematically derived from the $E_{11} \otimes_s l_1$ non-linear realisation. To derive this result we have only assumed that the equation is linear in derivatives and we have set to zero one constant; indeed the calculation is just a matter of E_{11} group theory. One cannot help be encouraged

by the very intricate way in which the invariance is achieved and it is a convincing sign, at least to this author, that this equation has almost all the correct features at low levels.

Additional references given in this note added

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